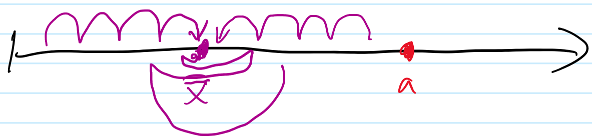
2.5b global stability

Monday, February 1, 2021 5:11 AM

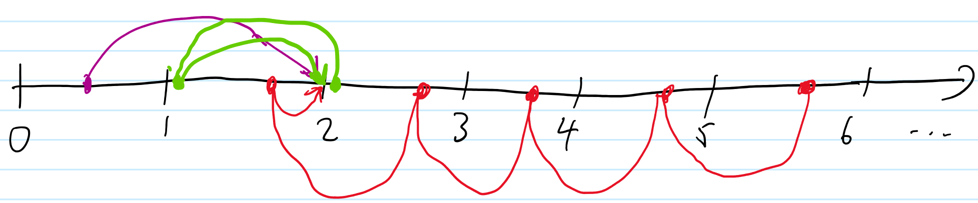
Def. 2.6 Suppose $\bar{x}$ is an equilibrium of the difference eq. $x_{t+1} = f(x_t)$, where $f: [0, a) \to [0, a)$, $0 < a \le \infty$. Then $\bar{x}$ is said to be **globally attractive** if $\forall\, x_0 \in (0, a)$, $\lim_{t\to\infty} x_t = \bar{x}$. The equilibrium $\bar{x}$ is **globally asymptotically stable** if it is globally attractive and locally stable.



Aside: If f is continuous, then globally attracting implies locally asymptotically stable.

Ex 2.7 Define the map $f: [0, \infty) \to [0, \infty)$:

$$f(x) = \begin{cases} 2 & , x \in [0, 2] \\ x-1 & , x \in (2, \infty) \end{cases}$$



Then for $x_{t+1} = f(x_t)$, $\forall\, x_0 \in [0, \infty)$, $\lim_{t\to\infty} x_t = 2$.

So, $\bar{x} = 2$ is globally attractive.

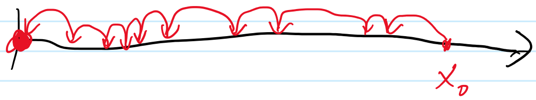
But, let $\varepsilon = \frac{1}{2}$. Then for $2 < x_0 < 2 + \varepsilon$, $x_1 < 1 + \varepsilon$ and $|x_1 - 2| > \frac{1}{2}$.

Thus, $\bar{x} = 2$ is not locally stable.

Thm 2.5 If $f: [0, a) \to [0, a)$ is continuous, and $0 < f(x) < x$ for all $x \in (0, a)$, then the origin is globally asymptotically stable.

proof. $\{x_0, x_1, \ldots\}$ is monotone decreasing and bounded by 0, and

proof. $\{x_0, x_1, \dots\}$ is monotone decreasing and bounded by 0, and 0 is also the only fixed pt by supposition. ☑

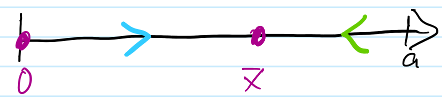


Thm 2.6 Let all of the following conditions hold:

(1) f is a continuous function on $[0, a)$, $0 < a \le \infty$

(2) $f: [0, a) \to [0, a)$, $0 < a \le \infty$

(3) $f(0) = 0$, $f(\bar{x}) = \bar{x}$.



(4) $f(x) > x$ for $0 < x < \bar{x}$

(5) $f(x) < x$ for $\bar{x} < x < a$

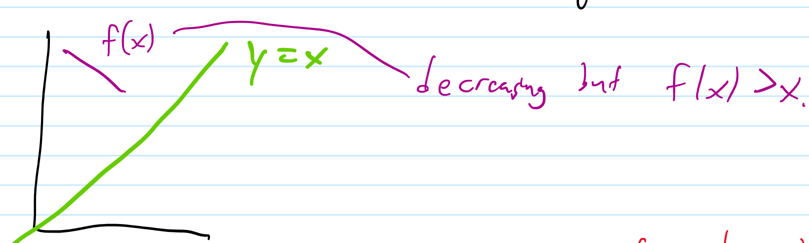
(6) If $\exists x_M \in (0, \bar{x})$ s.t. $f(x_M) = \sup_{x \in (0, \bar{x})} f(x)$, then f is decreasing for $x > x_M$.

(x_M is a maximum in $(0, \bar{x})$)

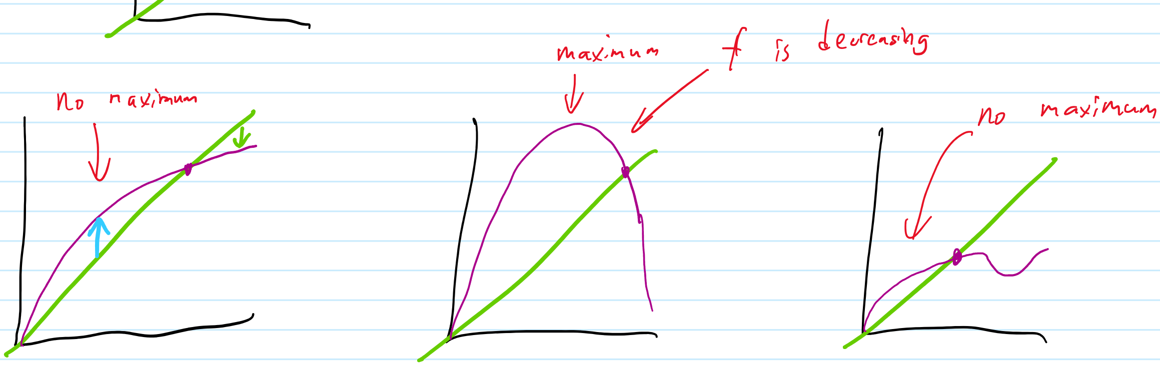
If not, cond 6 holds trivially.

When all conditions (1)–(6) hold, then $x_{t+1} = f(x_t)$ has a globally asymptotically stable equilibrium at $\bar{x}$ iff f has no 2-cycles.

Important note: f decreasing is NOT equiv. to $f(x) < x$



Ex.



Ex. 2.9 $$x_{t+1} = \frac{a x_t}{b + x_t}, \quad a > b > 0 \quad \text{on } [0, \infty)$$

Ex. 2.9 $x_{t+1} = \dfrac{a x_t}{b + x_t}$, $a > b > 0$ on $[0, \infty)$

Then (1/2) $f(x) = \dfrac{ax}{b+x}$, continuous on $[0, \infty)$

(3) $f(0) = 0$, $\bar{x} = a - b$. $f(a-b) = \dfrac{a(a-b)}{b + (a-b)} = a - b$

(4/5) $f(x) - x = \dfrac{ax}{b+x} - x = \underbrace{\dfrac{x}{b+x}}_{>0} [(a-b) - x] \begin{cases} > 0 & \text{if } x < a-b \\ < 0 & \text{if } x > a-b \end{cases}$

(6) $f'(x) = \dfrac{(b+x)a - ax}{(b+x)^2} = \dfrac{ab}{(b+x)^2} > 0$, so no maximum in $(0, \bar{x})$.

Further, suppose $x = f(f(x)) = \dfrac{a\left(\frac{ax}{b+x}\right)}{b + \frac{ax}{b+x}} = \dfrac{a^2 x}{b^2 + bx + ax}$

$$\underbrace{x = 0 \quad \text{or} \quad \begin{aligned} b^2 + bx + ax &= a^2 \\ x(a+b) &= a^2 - b^2 \\ x &= a - b \end{aligned}}_{\text{both are steady equilibria}}$$

Therefore, there are no 2-cycles, so $\bar{x} = a - b$ is a globally asymptotically stable solution.

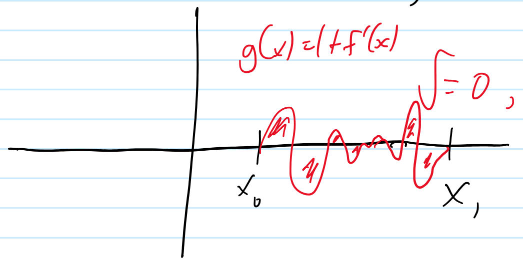
Thm 2.7 Let $f': I \to I$ be continuous. If $1 + f'(x) \neq 0 \ \forall x \in I$, then $x_{t+1} = f(x_t)$ has no 2-cycles in I.

proof. Suppose $\exists$ 2-cycle $\{x_0, x_1\}$ s.t. $f^2(x_0) = f(x_1) = x_0$, $x_0, x_1 \in I$.
WLOG, $x_0 < x_1$.

Then
$$\int_{x_0}^{x_1} (1 + f'(x))\, dx = (x_1 + f(x_1)) - (x_0 + f(x_0))$$
$$= f(x_0) + x_0 - x_0 - f(x_0) = 0.$$

Then by the MVT for integrals, $\exists c \in I$ s.t. $1 + f'(c) = 0$.

This is a contradiction. Hence, there are no 2-cycles in I.



$\int = 0$, then it must $=0$ at some pt

Ex. 2.10 $\quad x_{t+1} = \dfrac{a x_t}{b + (x_t)^k}$, $\quad a, b, k > 0$, $\quad x_0 > 0$.

$$f(x) = \frac{ax}{b + x^k}$$

$$1 + f'(x) = 1 + \frac{a(b + x^k(1-k))}{(b+x^k)^2}$$

If $k \le 1$, then $1 + f'(x) > 0$ for all $x > 2$, so no 2-cycles

Thm 2.8 If $f : [0, a) \longrightarrow [0, a)$ cont. and $\bar{x} \in (0, a)$ such that

$x < f(x) < \bar{x}$ for $0 < x < \bar{x}$

and $\bar{x} < f(x) < x$ for $x > \bar{x}$,

then the difference eq. $x_{t+1} = f(x_t)$ has a globally asymptotically stable equilibrium at $\bar{x}$.

proof. Note $x_t < x_{t+1} = f(x_t) < \bar{x}$ for $x_t < \bar{x}$

and $\bar{x} < x_{t+1} = f(x_t) < x_t$ for $x_t > \bar{x}$

Case 1: $x_0 < \bar{x}$. Then $\{ f^t(x_0) \}_{t=0}^{\infty}$ is monotone increasing and bounded above by $\bar{x}$.

Let $z_1 = \lim_{t\to\infty} f^t(x_0) = f\left(\lim_{t\to\infty} f^{t-1}(x_0)\right) = f(z_1)$, a fixed pt.

<u>Case 2:</u> $x_0 > \bar{x}$. Then $\{f^t(x_0)\}_{t=0}^{\infty}$ is monotone

decreasing and bounded below by $\bar{x}$

Let $z_2 = \lim_{t\to\infty} f^t(x_0) = f(z_2)$, a fixed pt.

But, the pos. fixed pt is $\bar{x}$, so $z_1 = z_2 = \bar{x}$. $\boxed{}$

<u>Ex.</u> $x_{t+1} = \dfrac{a x_t}{b + x_t} = f(x_t)$, $\quad a > b > 0$

$\bar{x} = a-b$. $\qquad \dfrac{ax}{b+x} - x = \dfrac{x}{b+x}\left[(a-b) - x\right] \begin{cases} > 0 & \text{if } 0 < x < a-b \\ < 0 & \text{if } x > a-b \end{cases}$

$$\frac{ax}{b+x} - (a-b) = \frac{ax - ab - ax + b^2 + bx}{b+x} = \frac{b}{b+x}\left[x - (a-b)\right]$$

$$\begin{cases} < 0 & \text{if } 0 < x < a-b \\ > 0 & \text{if } x > a-b. \end{cases}$$

So, by Thm 2.8, $\bar{x} = a-b$ is globally asymptotically stable.

<u>Thm 2.9</u> Let $x_{t+1} = f(x_t)$

(1) f is a continuous function on $[0, a)$, $0 < a \le \infty$

(2) $f : [0, a) \to [0, a)$, $\quad 0 < a \le \infty$

(3) $f(0) = 0$, $f(\bar{x}) = \bar{x}$.

(4) $f(x) > x$ for $0 < x < \bar{x}$

(5) $f(x) < x$ for $\bar{x} < x < a$

(6) If $\exists x_M \in (0, \bar{x})$ s.t. $f(x_M) = \sup f(x)$, (x_M is a)

(5) $f(x) < x$ for $\bar{x} < x < a$

(6) If $\exists x_M \in (0, \bar{x})$ s.t. $f(x_M) = \sup_{x \in (0,\bar{x})} f(x)$, then f is decreasing for $x > x_M$.

(x_M is a maximum in $(0, \bar{x})$)

(a) Suppose f satisfies assumptions (1)–(5) but f has no maximum in $(0, \bar{x})$. Then $\bar{x}$ is globally asymptotically stable.

(b) Suppose f satisfies assumptions (1)–(6) and f has a maximum x_M in $(0, \bar{x})$. Then $\bar{x}$ is globally asymptotically stable iff $f(f(x)) > x$ for all $x \in [x_M, \bar{x})$.

Ex. 2.12 $x_{t+1} = 2 x_t e^{-r x_t} = f(x_t)$, $r > 0$.

Two equilibria at 0 and $\bar{x} = \dfrac{\ln 2}{r}$.

Can check conditions (1)–(5) are satisfied.

Also
$$f(x) = 2x e^{-rx}$$
$$f'(x) = 2e^{-rx} - 2rx e^{-rx}$$
$$f'(x) = 2(1 - rx) e^{-rx}$$

Setting $f'(x_M) = 0 = 2(1 - r x_M) e^{-r x_M}$

$\Rightarrow x_M = \frac{1}{r}$ is a local max.

But $\frac{1}{r} > \frac{\ln 2}{r}$, so f has no max in $(0, \bar{x})$.

Thus, $\bar{x}$ is globally asymptotically stable.